Period 3, Jan 8, 2025

1. Use the graph of $y = f(x)$ on the closed interval [0,6] below to find the following:

a) The open intervals on which f is increasing:

$(1,3) \cup (4,6)$

b) The open intervals on which f is decreasing:

$(0,1) \cup (3,4)$

c) The open intervals on which f is concave downward:

$(2,4) \cup (4,6)$

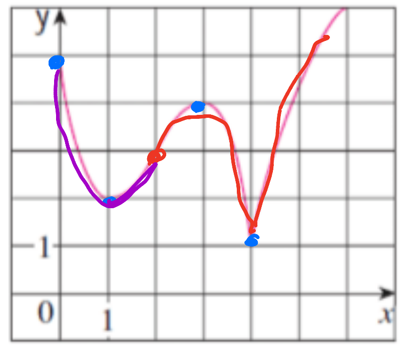
d) The open intervals on which f is concave upward:

$(0,2)$

e) The coordinates of the points of inflection:

$(2,3)$

concavity must change



3. Given $f(x) = \sin x - \cos x$. Find the Point(s) of inflection of f(x) on $[0, 2\pi]$. Justify your answer.

$F'(x) = \cos x - (-\sin x)$

$F'(x) = \cos x + \sin x$

$F''(x) = -\sin x + \cos x$

$0 = -\sin x + \cos x$

$\sin x = \cos x$

$= \frac{\pi}{4}, \frac{5\pi}{4}$

$F''(x) = 0$ or $\varnothing$

$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad$ $\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad$ $\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

5. Use the Second Derivative Test to find the relative extrema of

$h'(x) = 0$ or $\varnothing$

a) $f(x) = 4x + \frac{4}{x}$.

b) $h(x) = 6x - x^2$

$h'(x) = 6 - 2x$

$h''(x) = -2$

Concave Down Everywhere $\to$ Max

$0 = 6 - 2x$

$3 = x$

$h(3) = 6(3) - 3^2 = 18 - 9 = 9$

$(3,9)$

6. If $f(t) = 2t^3 + 3t^2 - 36t$, find

(a) the intervals on which f is increasing or decreasing. Justify your response.

$F'(x) = -$

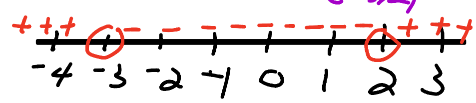
increase $(-\infty, -3) \cup (2, \infty)$

decrease $(-3, 2)$

$F'(-4) = 6(-4+3)(-4-2) = +$ $F'(x) = +$

$F'(0) = 6(0+3)(0-2) = + \cdot + \cdot - = -$

$F'(3) = 6(3+3)(3-2) = + \cdot + \cdot + = +$



(b) the t-values of local maximum and minimum values. Justify your response.

$X = -3$ and $X = 2$

concave Down Max

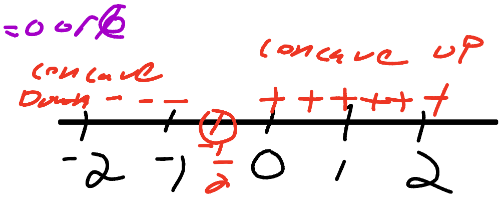
concave up = Min

$F(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = 81$

$F(2) = 2(2)^3 + 3(2)^2 - 36(2) = -44$

(c) the intervals of concavity and inflection points. Justify each response.

$F''(x) = 0$ or $\cancel{0}$



$F(x) = 2x^3 + 3x^2 - 36x$

$F'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x+3)(x-2)$

$F'(-3) = 6(-3+3)(-3-2) = 6 \cdot 0 \cdot -5 = 0$

$F'(2) = 6(2+2)(2-2) = 6 \cdot 4 \cdot 0 = 0$

$F''(x) = 12x + 6$

$F''(-2) = 12(-2) + 6 = -24 + 6 = -$

$F''(x) = 0$

$F''(0) = 12(0) + 6 = 0 + 6 = +$

$X = -\frac{1}{2}$

7. If $g(x)=\frac{x^2}{x^2-1}$, find

$(-\infty,-1)\cup(-1,0)$ $\quad$ $(0,1)\cup(1,\infty)$

$$g'(x)=\frac{-2x}{(x^2-1)^2}$$

(a) the intervals on which g is increasing or decreasing. Justify your response.

$+ + + + + + + 0 + + - - - 0 - - - - - -$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

$g(-3)=\frac{-2(-3)}{+}=\frac{+}{+}=+$

$g(-\frac{1}{2})=\frac{-2(-\frac{1}{2})}{+}=\frac{+}{+}=+$

$g(\frac{1}{2})=\frac{-2\cdot\frac{1}{2}}{+}=\frac{-}{+}=-$

$g(3)=\frac{-}{+}=-$

(b) the x-values of local maximum and minimum values. Justify your response.

$x=0$
$x=1$
$x=-1$

$g(1)=\emptyset$
$g(-1)=\emptyset$
$g(0)=\frac{0^2}{0^2-1}=\frac{0}{-1}=0$

(c) the intervals of concavity and inflection points. Justify your response.

Points of inflection $x=-1$, $x=1$

$g(1)=\emptyset$
$g(-1)=\emptyset$

$+ + + + + + + \; - - - - - \; + + + + + + +$

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

$$g''(x)=\frac{6x^2+2}{(x^2-1)^3}$$

$g''(-2)=\frac{6(-2)^2+2}{((-2)^2-1)^3}=\frac{+}{+}$

$g''(0)=\frac{6(0)^2+2}{(0^2-1)^3}=\frac{+}{-}$

$g''(2)=\frac{6(2)^2+2}{(2^2-1)^3}=\frac{+}{+}$

$$g(x)=\frac{x^2}{x^2-1}=\frac{x^2}{(x-1)(x+1)}$$

$g(1)=\emptyset$
$g(-1)=\emptyset$

$$g'(x)=\frac{2x(x^2-1)-x^2(2x)}{(x^2-1)^2}=\frac{\cancel{2x^3}-2x-\cancel{2x^3}}{(x^2-1)^2}=\frac{-2x}{(x^2-1)^2}\Rightarrow g'(x)=0 \text{ or } \emptyset$$

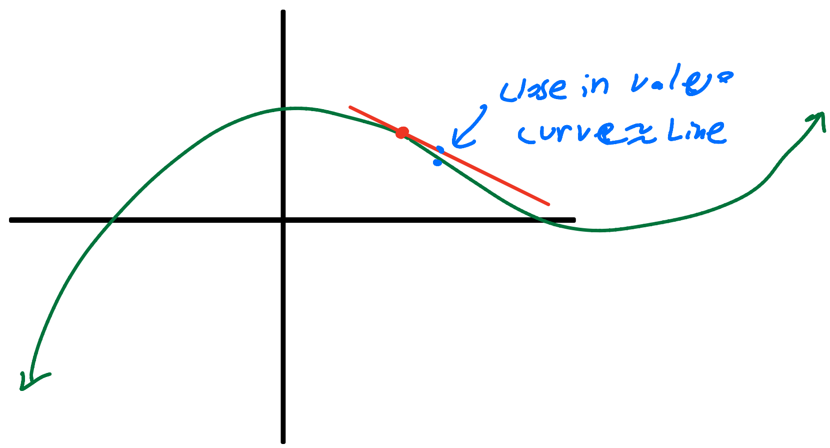
$x=0,1,-1$

$$g''(x)=\frac{-2(x^2-1)^2-(-2x)(2(x^2-1)^1\cdot 2x)}{((x^2-1)^2)^2}=\frac{\cancel{(x^2-1)}[-2(x^2-1)+8x^2]}{(x^2-1)^{\cancel{4}\,3}}$$

$-2x^2+2+8x^2$

$$g''(x)=\frac{6x^2+2}{(x^2-1)^3}=\frac{\text{always }+}{(x-1)^3(x+1)^3}$$

$x=1$
$x=-1$ $\Big\}$ Possible inflection PTS



Example 1: Find the tangent line approximation of $f(x) = 1 + \sin x$ at the point (0, 1). Then use a table to compare the y-values of the linear function with those of $f(x)$ on an open interval containing $x = 0$.

$F(x) = 1 + \sin x$

$F'(x) = \cos x$

$(0,1) \rightarrow$ point

$F'(0) = \cos 0 = 1 =$ Slope

$y - 1 = 1(x - 0)$

$y - 1 = x$

$y = x + 1$

$y = 1 + \sin x$

$\approx$

$y = x + 1$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

as $x \to 0$ $\sin x \approx x$

Example 2

Use local linearization to approximate $\sqrt{16.5}$. Is your answer and overestimate or an underestimate. Justify your reasoning.

$y = \sqrt{x} = x^{\frac{1}{2}}$

Found a Function That matches

$\sqrt{16} = 4$

$\downarrow .5$

$\sqrt{16.5} \approx$

Tangent Line aT (16,4) PoinT

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$\text{Slope} = \frac{1}{2\sqrt{16}} = \frac{1}{8}$

$y = \frac{1}{8}x + b$

$4 = \frac{1}{8} \cdot 16 + b$

Found Tangent Line

$4 = 2 + b$

$2 = b$

$y = \frac{1}{8}x + 2$

$$
\begin{array}{r}
2.0625 \\
8 \overline{)16.5000} \\
\underline{16} \\
0\ 50 \\
\underline{48} \\
20 \\
\underline{16} \\
40
\end{array}
$$

$\sqrt{16.5} \approx \frac{1}{8}(16.5) + 2$

$2.0625 + 2$

$4.0620192 \approx 4.0625$

Solved

$3^3 = 27$ → Find Tangent Line at $X=3$

Your Turn: Estimate $(3.01)^3$ without a calculator. Is your answer and overestimate or underestimate? Justify your answer.

$y = x^3$

Point $(3,27)$

$F'(x) = \frac{dy}{dx} = 3x^2$

$F'(3) = 3(3)^2 = 27 = \text{Slope}$

$y = 27x + b$

$27 = 27(3) + b$

$27 = 81 + b$

$-54 = b$

$y = 27x - 54$ ← Tangent Line

$y = 27(3.01) - 54$ Plug in $X = 3.01$

$81.27 - 54$

27.27

$F'(x) = 3x^2$

$F''(x) = 6x = 6(3.01) = +$ concave up Under estimate

$\frac{dy}{dx} = \frac{\text{change in } y}{\text{change in } x}$

$\cancel{dx} \cdot \frac{dy}{\cancel{dx}} = 3x^2 \cdot dx = 3(3)^2(.01) = .27$

$dy = 3x^2 \cdot dx$

Point $(3,27)$

$dx = .01$ ↓ ↓ $dy = 0.27$

$x = 3$ $(3.01, 27.27)$

The number of gallons, $P(t)$, of a pollutant in a lake changes at the rate $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ gallons per day, where t is measured in days. There are 50 gallons of the pollutant in the lake at time $t = 0$. The lake is considered to be safe when it contains 40 gallons or less of pollutant.

(d) An investigator uses the tangent line approximation to $P(t)$ at $t = 0$ as a model for the amount of pollutant in the lake. At what time t does this model predict that the lake becomes safe?

$P'(0) = 1 - 3e^{-0.2\sqrt{0}} = 1 - 3e^{0} = 1 - 3 \cdot 1 = -2 =$ Slope

Point $(0, 50)$

$y = -2x + b \Rightarrow$ Tangent Line

$y = -2x + 50$

$40 > -2x + 50$

$-50 \qquad -50$

$-10 > -2x$

$\overline{-2} \quad \overline{-2}$

$5 < x$

Let f be a function for which $f(2) = 6$ and $f'(2) = -3$.

Point $(2, 6)$ Slope $= -3$

If the tangent line to a graph of f at 2 is used to approximate a zero of f, then the approximation is

$y = mx + b$

$y = -3x + b$

$6 = -3(2) + b$

$6 = -\cancel{6} + b$

$+6 \quad +\cancel{6}$

$12 = b$

$y = -3x + 12$

$0 = -3x + 12$

$x = 4$

$y - y_1 = m(x - x_1)$

$y - 6 = -3(x - 2)$

$y - \cancel{6} = -3x + 6$

$+\cancel{6} \quad +6$

$y = -3x + 12$

$\frac{\pi}{2} \cdot x$

A linear approximation to $f(x) = x \cdot \sin\left(\frac{\pi x}{2}\right) + x^2$ at $x = 3$ is

$F(3) = 3\sin\frac{3\pi}{2} + 3^2 = 3 \cdot -1 + 9 = 6$

$(3, 6)$

$F'(x) = 1 \cdot \sin\frac{\pi x}{2} + x \cdot \left(\cos\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right) + 2x$

$F'(3) = 1 \cdot -1 + 3 \cdot 0 \cdot \frac{\pi}{2} + 2 \cdot 3 = 5 = \text{Slope}$

$y = 5x - 9$

If $f'(x) = 2xe^{x^2-1} - 3\pi\sin(\pi x)$ and $f(1) = 4$, approximate $f(1.03)$ using a linear approximation.

Slope = 2

Point (1, 4)

$y = 2x + b$

$y = 2x + 2$

$F(1.03) \approx 2(1.03) + 2$

$F(1.03) \approx 4.06$

Example 3: Find the derivative of each function in differential form.

a) $y = 2t^3 + 5t^2 - 3t + 1$

$$\frac{dy}{dT} = 6T^2 + 10T - 3$$

$$dy = (6T^2 + 10T - 3)\,dT$$

Example 5: Inflating a bicycle tire changes its radius from 12 inches to 13 inches. Use differentials to estimate the change in perimeter of the tire.

$r = 12 \rightarrow r = 13$

Radius starts at 12 inches so $r = 12\ inches$

$dr = 1$

$C = 24\pi$

Radius goes from 12 to 13 so $dr = 1\ inch$

$C = 26\pi$

$C = 2\pi r$

$$\frac{dC}{dr} = 2\pi$$

$dC = 2\pi\, dr$

$dC = 2 \cdot \pi \cdot 1$

$dC = 2\pi$